

Estimation and Optimal Filtering for a Common Stochastic Cycle Shifted in Continuous Time

Stima e filtro ottimale per l'estrazione di un ciclo stocastico comune ritardato in tempo continuo

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Riassunto: Nell'analisi delle serie storiche le relazioni di anticipo o ritardo tra variabili vengono comunemente modellate per mezzo di ritardi multipli della frequenza di campionamento. Questo approccio è sub-ottimale quando il fattore comune può essere pensato come un processo a tempo continuo che si manifesta con ritardi diversi su diverse serie temporali osservate ad istanti discreti di tempo. La situazione descritta è quella tipica dell'analisi del ciclo economico e, pertanto, in questo lavoro si mostra come stimare i parametri continui di ritardo di un ciclo stocastico comune a più serie e come estrarre il relativo segnale in maniera L^2 -ottimale.

Keywords: stochastic cycle, phase shifts, Kalman filter, unobserved components, structural time series models, business cycle.

1. Introduction

It is quite common in business cycle analysis to build a coincident indicator (CI) through a set of variables that are considered contemporaneous to the business cycle and one or more leading (lagging) indicators that are expected to anticipate (delay) the movement of the CI. In doing this analysis, though, only integer delay parameters are allowed, and the estimation of the delay is usually carried out by means of visual inspection of the sample cross-correlations between pairs of (often differenced) time series. The integer delay constrain may be sometimes considered a reasonable approximation, but an exact method should give more accurate results both in the extraction of the business cycle signal and in its prediction through leading variables. Furthermore, a growing literature on the convergence of business cycles of different countries requires methods for allowing the relative phase parameters to be (smoothly) time varying: this is only possible with delay parameters taking values in $\mathbb{R}$.

In a recent work Rünstler (2004) has proposed a brilliantly easy way to model phase shifts among stochastic cycles and his method has been implemented by Valle e Azevedo *et al.* (2004), Koopman and Valle e Azevedo (2004) and Pelagatti (2005). However, Rünstler's method, which consist of a rotation of the cycle by a fixed angle, does not allow the modelling of the simple delay filter $\phi_t = \psi_{t+k}$, where ψ_t is the (usually unobservable) stochastic cycle and $k \in \mathbb{R}$.

In this short communication, we develop a model for estimating the continuous time delay between two time series with common stochastic cycle of the type used in structural time series models (cfr. Harvey, 1989). Due to space limitations, we address the problem

only for two stock variables and leave the generalization to n time series of stock and flow variables for a longer communication.

2. The model

Consider the following *delayed common stochastic cycle plus noise model*

$$\begin{aligned} y_t &= \psi_t + \varepsilon_t \\ z_t &= \beta \phi_t + \eta_t \end{aligned} \quad (1)$$

with ε_t and η_t orthogonal white noise disturbances, $\phi_t = \psi_{t+k}$, common stochastic cycle, shifted by a time lag of $k \in \mathbb{R}^+$ periods and β scale coefficient.

Let the process ψ_t be the continuous time stochastic cycle discussed by Harvey (1989, ch.9), which in “differential form” (cfr. Bergstrom, 1984) is given by

$$\frac{d}{dt} \begin{bmatrix} \psi(t) \\ \psi^*(t) \end{bmatrix} = \begin{bmatrix} \log \rho & \lambda \\ -\lambda & \log \rho \end{bmatrix} \begin{bmatrix} \psi(t) \\ \psi^*(t) \end{bmatrix} + \begin{bmatrix} \kappa_c(t) \\ \kappa_c^*(t) \end{bmatrix}, \quad (2)$$

where $\kappa_c(t)$ and $\kappa_c^*(t)$ are orthogonal continuous time white noise processes with common variance σ_c^2 . The exact discrete time representation of (2) for any positive time span δ is

$$\begin{bmatrix} \psi_{t+\delta} \\ \psi_{t+\delta}^* \end{bmatrix} = \rho^\delta \begin{bmatrix} \cos \lambda \delta & \sin \lambda \delta \\ -\sin \lambda \delta & \cos \lambda \delta \end{bmatrix} \begin{bmatrix} \psi_t \\ \psi_t^* \end{bmatrix} + \begin{bmatrix} \kappa_t \\ \kappa_t^* \end{bmatrix}, \quad (3)$$

or more synthetically, with self-evident notation,

$$\psi_{t+\delta} = \rho^\delta \mathbf{T}_{\lambda\delta} \psi_t + \kappa_t, \quad (4)$$

where κ_t and κ_t^* are white noise processes obtained by integrating $\kappa_c(t)$ and $\kappa_c^*(t)$ over the interval $(t, t + \delta]$ or, for stationarity, over $(0, \delta]$, with covariance matrix

$$\Sigma_\kappa = \sigma_\kappa^2 \mathbf{I}_2 = \frac{\sigma_c^2(1 - \rho^{2\delta})}{\log \rho^{-1}} \mathbf{I}_2. \quad (5)$$

For $\delta = 1$, equation (3) becomes the usual stochastic cycle used in structural time series models.

Now, the stochastic cycle at time $t + h + \delta$, with h non-negative integer and $\delta \in [0, 1)$, is the first element of the vector

$$\psi_{t+h+\delta} = \rho^\delta \mathbf{T}_{\lambda\delta} \psi_{t+h} + \nu_{t+h}(\delta), \quad (6)$$

where $\nu_t(\delta)$ is a bivariate white noise with covariance matrix

$$\Sigma_\nu(\delta) = \sigma_\nu^2(\delta) \mathbf{I}_2 = \frac{\sigma_c^2(1 - \rho^{2\delta})}{\log \rho^{-1}} \mathbf{I}_2 = \frac{1 - \rho^{2\delta}}{1 - \rho^2} \sigma_\kappa^2 \mathbf{I}_2.$$

Notice that the covariance of $\nu_{t+h}(\delta)$ with κ_{t+h} is given by

$$\begin{aligned} \sigma_{\nu\kappa}(\delta) &= E(\nu_{t+h}(\delta) \kappa_{t+h}) = E\left(\int_0^\delta \kappa(r) dr \int_0^1 \kappa(r) dr\right) \\ &= E\left(\int_0^\delta \kappa(r) dr\right)^2 + E\left(\int_0^\delta \kappa(r) dr \int_\delta^1 \kappa(r) dr\right) \\ &= E(\nu_t(\delta)^2) + 0 = \sigma_\nu^2(\delta) \end{aligned} \quad (7)$$

and the same is true for $\nu_{t+h}^*(\delta)$ and κ_{t+h}^* (obviously the covariance between ‘star-elements’ and ‘non-star-elements’ is zero). Thus, under Gaussianity, $\nu_{t+h}(\delta)|\kappa_{t+h}$ is a normal random variable with mean

$$E(\nu_{t+h}(\delta)|\kappa_{t+h}) = \frac{\sigma_{\nu\kappa}(\delta)}{\sigma_\kappa^2} \kappa_{t+h} = \frac{1 - \rho^{2\delta}}{1 - \rho^2} \kappa_{t+h}, \quad (8)$$

and variance

$$\text{Var}(\nu_{t+h}(\delta)|\kappa_{t+h}) = \sigma_\nu^2(\delta) - \frac{\sigma_{\nu\kappa}^2(\delta)}{\sigma_\kappa^2} = \sigma_\kappa^2 \left[\frac{(1 - \rho^{2\delta})(\rho^{2\delta} - \rho^2)}{(1 - \rho^2)^2} \right]. \quad (9)$$

If the system is not normal (8) is a linear projection, but no more a conditional expectation.

Using backward substitutions we can write the evolution of the coincident and leading cycles with respect to time $t - 1$:

$$\psi_t = \rho \mathbf{T}_\lambda \psi_{t-1} + \kappa_{t-1} \quad (10)$$

$$\psi_{t+h} = \rho^{h+1} \mathbf{T}_\lambda^{h+1} \psi_{t-1} + \kappa_{t+h-1} + \rho \mathbf{T}_\lambda \kappa_{t+h-2} + \dots + \rho^h \mathbf{T}_\lambda^h \kappa_{t-1}, \quad (11)$$

and the transition from time $t + h$ to time $t + h + \delta$ is given by equation (6).

Now we have all the elements needed to put the model in state-space form and carry out estimation and filtering by means of the Kalman filter and maximum likelihood:

observation equations

$$\begin{aligned} y_t &= \mathbf{e}_1' \psi_t + N(0, \sigma_y^2) \\ z_t &= \beta [\rho^\delta \boldsymbol{\tau}'_{\delta\lambda} \phi_t + \nu_t] + N(0, \sigma_z^2) \end{aligned}$$

transition equations

$$\begin{aligned} \kappa_t^{(0)} &= N_2(\mathbf{0}, \sigma_k^2 \mathbf{I}_2) \\ \kappa_t^{(1)} &= \kappa_{t-1}^{(0)} \\ &\dots \\ \kappa_t^{(h+1)} &= \kappa_{t-1}^{(h)} \\ \psi_t &= \rho \mathbf{T} \psi_{t-1} + \kappa_{t-1}^{(h+1)} \\ \omega_t &= \kappa_{t-1}^{(0)} + \rho^1 \mathbf{T}_\lambda^1 \kappa_{t-1}^{(1)} + \dots + \rho^h \mathbf{T}_\lambda^h \kappa_{t-1}^{(h)} \\ \phi_t &= \rho^{h+1} \mathbf{T}_\lambda^{h+1} \psi_{t-1} + \omega_{t-1} \\ \nu_t &= \frac{(1 - \rho^{2\delta})}{(1 - \rho^2)} \kappa_{t-1}^{(0)} + N\left(0, \sigma_\kappa^2 \frac{(1 - \rho^{2\delta})(\rho^{2\delta} - \rho^2)}{(1 - \rho^2)^2}\right) \end{aligned}$$

where h is a nonnegative integer, δ takes values in the real unit interval, β is a real loading coefficient, $\boldsymbol{\tau}'_{\delta\lambda} = [\cos \delta\lambda \ \sin \delta\lambda]$ is the first row of $\mathbf{T}_{\delta\lambda}$ and $\mathbf{e}_i$ is the i -th column of the two-dimensional identity matrix. Let $k \in \mathbb{R}$ be the continuous time delay parameter:

- (i) if $k > 0$ then put $h = \lfloor k \rfloor$ and $\delta = k - h$, where $\lfloor k \rfloor$ is the integer part of k ;
- (ii) if $k = 0$ then the representation reduces to that of a standard common cycle model;
- (iii) if $k < 0$ then put $h = \lfloor |k| \rfloor$, $\delta = |k| - h$ and change the observation equations to

$$\begin{aligned} y_t &= \rho^\delta \mathbf{T}_1^\delta \phi_t + \nu_t + N(0, \sigma_y^2) \\ z_t &= \beta \mathbf{e}_1' \psi_t + N(0, \sigma_z^2). \end{aligned}$$

A practical problem in efficiently running the Kalman filter (and smoother) is encountered when h is large, since the transition matrix dimensions grow with the square of $2(h + 2)$. A reduction of the state vector can be accomplished by using the unbalanced stochastic cycle discussed in Harvey and Trimbur (2003) defined by

$$\begin{pmatrix} \psi_t \\ \psi_t^* \end{pmatrix} = \rho \begin{pmatrix} \cos \lambda & \sin \lambda \\ -\sin \lambda & \cos \lambda \end{pmatrix} \begin{pmatrix} \psi_{t-1} \\ \psi_{t-1}^* \end{pmatrix} + \begin{pmatrix} \kappa_{t-1} \\ 0 \end{pmatrix} \quad (12)$$

where κ_{t-1} has the same properties as above. The stochastic properties of this cycle are very similar to those of the balanced cycle, but the transition matrix dimensions of the state space form increase only with the square of $(h + 2)$. Furthermore, if algorithms for the efficient computation of operations on sparse matrices are implemented, as in the about to be released version 3 of SsfPack by Koopman *et al.* (1999), then the dimensions of the transition matrix become less worrying, being its nonzero elements quite few.

The joint estimation of the continuous time delay and of the other (hyper-) parameters of the model can be carried out by standard prediction error decomposition of the likelihood Harvey (1989, §3.4). The state space form of the model changes its dimensions according to the value of the continuous time delay parameter $k = h + \delta$, but this does not involve jumps in the likelihood function. Indeed, it can be easily checked that the observation equations are continuous functions of k . There is no need to show that a trend or a seasonal component may be added to the state space representation of our model with negligible additional effort.

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